MR1212253 (94g:32043) 32S25 (14B15 32C36 32S40)
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Quotient spaces and critical points of invariant functions for $\mathbf{C}^*$-actions.
J. Reine Angew. Math. **437** (1993), 55–99.

Consider a linear $\mathbf{C}^*$-action on the smooth germ $X=(\mathbf{C}^{n+1},0)$. This action is completely determined by a collection of $n+1$ integer weights defined from its canonical diagonal form. Let a,b,c be the numbers of positive, negative and zero weights, respectively. Denote by $\lambda=(\lambda_1,\cdots,\lambda_a)$ the positive and by $-\mu=(-\mu_1,\cdots,-\mu_b)$ the negative weights of the action. The authors prove that the quotient germ Y can be presented as the direct product $Y_0\times F$, where Y_0 is the weighted cone over the direct product $\mathbf{P}^{a-1}_\lambda\times\mathbf{P}^{b-1}_\mu$ of weighted projective spaces and F is a c-dimensional smooth germ. Let $(\Omega^\bullet_X,d)$ be the de Rham complex of regular holomorphic differential forms on X. Denote by ξ the Euler vector field generating the $\mathbf{C}^*$-action, by ι_ξ the contraction along ξ and by $L_\xi=\iota_\xi d+d\iota_\xi$ the Lie derivative. The authors calculate the local cohomology groups with support in $\{0\}\subset Y$ of the $\mathcal{O}_Y$-modules $\underline{\Omega}^p_X=\{\omega\in\Omega^p_X\colon\ L_\xi(\omega)=0\}$ and $\underline{\Omega}^p_\xi=\operatorname{Ker}\{\iota_\xi\colon\ \underline{\Omega}^p_X\to\underline{\Omega}^{p-1}_X\}$ for $p\geq 0$.

Let $f\colon X\to\mathbf{C}$ be the germ of an analytic function that is invariant under the $\mathbf{C}^*$-action. Thus, f can be considered as a function germ $f\colon Y\to\mathbf{C}$. Under our assumptions the 1-form df belongs to $\underline{\Omega}^1_X$ and $\underline{\Omega}^1_\xi$. Therefore the two complexes of $\mathcal{O}_Y$-modules $(\underline{\Omega}^\bullet_X,df\wedge)$ and $(\underline{\Omega}^\bullet_\xi,df\wedge)$ are well defined. The authors compute the cohomology of these complexes in the case where f has an isolated critical point at the origin $\{0\}\subset Y$. They prove that by analogy with the case of a function with isolated critical point on a smooth germ the dimension of $H^{n+1}(\underline{\Omega}^\bullet_X,df\wedge)\cong\underline{\Omega}^{n+1}_X/df\wedge\underline{\Omega}^n_X$ may be considered as a multiplicity of the critical point, although there are some cases where f has a critical point but the multiplicity is equal to zero. This multiplicity, like all dimensions of the lower cohomology groups, behaves well under a deformation of f. An explicit expression for the Gauss-Manin connection associated with a 1-parameter deformation of such a function is obtained. It turns out that this connextion is regular singular. Properties of a function invariant under real or symplectic $\mathbf{C}^*$-actions are discussed in detail.

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